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MUSAC II is a mathematical representation of passive sonar classification. The principle attribute of MUSAC II is its multiple target capability. The methodology is based on the detection of Lofar lines, Demon lines, and broadband noise. MUSAC II uses a sequential, Bayesian decision-making approach. The methodology is designed for use in a Monte Carlo, time step simulation of acoustic warfare engagements.

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MUSAC II

A Method for Modeling Passive Sonar Classification in a Multiple Target Environment

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PREFACE

The MUSAC II methodology described in this technical note is a revision of the original MUSAC model described in the draft report: "MUSAC--A Representation of the Passive Sonar Classification Process," SRI Project 1318-240, November 1973. The major revisions to the methodology include a new formulation of a multifeature sonar detection model, different likelihood calculations, and a more generalized decision making procedure. This technical note provides a concise theoretical description of the MUSAC II methodology; the new concepts have not, as yet, been tested by a computer implementation of the methodology.

The purpose of the research was to create a methodology that mathematically represents the passive sonar classification process in a multiple target environment. The MUSAC II methodology is not intended to be a software package for real classification hardware; instead, MUSAC II is intended to be used by analysts to study passive sonar systems. The primary application of the MUSAC II methodology will be for detailed Monte Carlo simulation modeling of acoustic warfare engagements. The methodology will provide classification decisions that can be used to initiate tactics in an engagement model. The MUSAC II methodology uses the standard acoustic parameters of classical sonar detection theory. By using a physical-based approach, the methodology can represent the inherent classification capability of a sonar system, particularly the sensitivity to signal-to-noise ratios. The alternative approach would try to duplicate the man/machine classification process by simulating the human perception of classification clues. MUSAC II does not try to duplicate the man/machine classification process, instead MUSAC II determines classification decisions from the fundamental information content of the acoustics. MUSAC II represents an ideal sonar classifier in the same sense that classical theory represents an ideal sonar detector. The detection capability of detection models can be adjusted to simulate real systems; in the same way, the classification capability of MUSAC II can be adjusted to simulate real passive sonar systems.

The MUSAC II project was conceived as a continuation of a classification model development effort under the sponsorship of James G. Smith, Code 431, Office of Naval Research. As part of an ONR reorganization, the MUSAC II project was transferred to Code 230. After evaluating the potential application and merit of the methodology relative to Code 230 program objectives, the project was redirected to tasks involving tactical development and evaluation research. This technical note reports on the partially completed research of the original tasking.

The authors are indebted to G. W. Black and W. F. Frye who were the originators of the basic ideas of the MUSAC methodology.

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I INTRODUCTION

The Multiple Source Acoustic Classification (MUSAC) methodology is a mathematical representation of passive sonar classification. The principal attribute of the MUSAC II methodology is its multiple-target capability. Almost without exception, other models allow for only one target at a time. The methodology is based on the detection of acoustic features. In this way, spectral and spatial acoustic information is included so that the sonar systems' bearing and frequency resolution can be related to the classification outcome. The acoustic features are defined by the analyst; the features can be narrowband, broadband, or modulated broadband classification clues (for example, Lofar or Demon lines). The acoustic features are represented by Bernoulli random variables. The stochastic structure of the model provides for realistic random variations of acoustic data. A dynamic encounter is represented by a time-step simulation. The MUSAC II methodology is structured for sequential decision-making by the update of classification information and the change in kinematic variables over time. From Monte Carlo replications, the probability of making selected tactical and classification decisions can be estimated.

The MUSAC II methodology uses a Bayesian decision-making approach. Figure 1 shows the model flow. The analyst first formulates a set of multiple-target hypotheses that will be used in the engagement simulation. The probability of detecting specified acoustic features is calculated at each time step, for each sonar look angle, and for each hypothesis (the true target configuration is usually one of the hypotheses). These detection probabilities are then used, in conjunction with the observed random features, to calculate the likelihood that the data would be observed if the hypothesis were true. The likelihoods and the prior probabilities are then combined to produce the posterior probability that the ith hypothesis is true, given the observed data. The analyst defines tactical or classification decisions that are to be simulated, he defines the value of making the decision when each hypothesis is true, and he defines value thresholds. With this decision structure, MUSAC II determines if a decision is to be made at the present time step; if not, another time-step is simulated and more data collected. If a decision is made, the decision with the largest average value is chosen. The above brief outline of the MUSAC II methodology is discussed in detail in the four following chapters, as indicated on Figure 1.

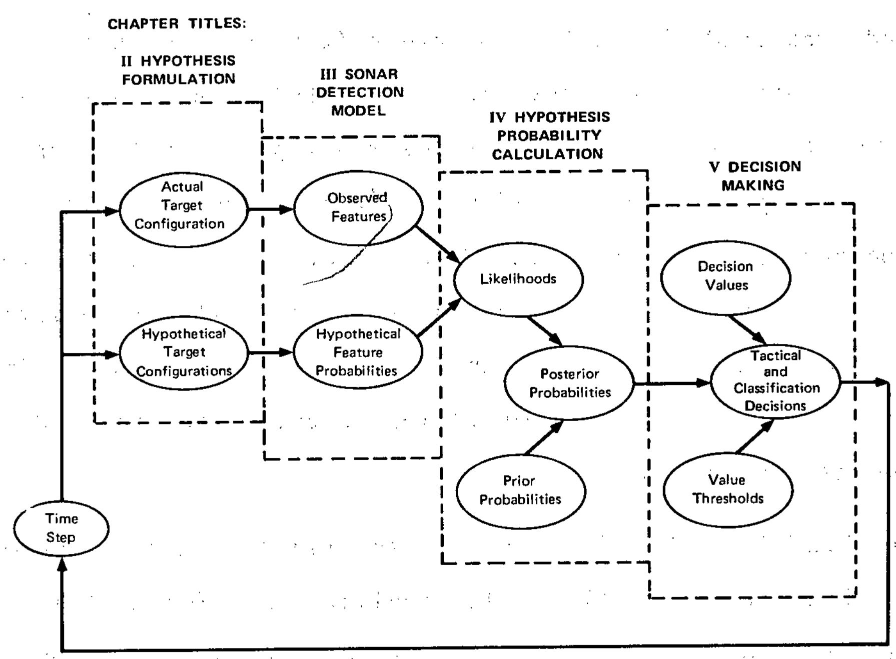

FIGURE 1 MUSAC II MODEL FLOW

II HYPOTHESIS FORMULATION

Hypothesis formulation is the most critical part of the methodology because the hypothesis set directly determines the possible target group configurations under consideration, and because the size of the hypothesis set determines the computational burden required to execute the model. The formulation of multitarget hypotheses are unique to MUSAC II and are very powerful in that they allow the multitarget classification problem to be addressed.

The multitarget hypothesis set consists of ordered arrangements of two components: single-target classes and target tracks.

A. Single-Target Classes

The set of single-target classes is a list of target types that the observer expects to encounter. For any particular application of MUSAC II, the set of single-target classes is defined by the analyst. Depending on the problem at hand, the single-target classes can be very general (submarine, high value warship, low value warship, merchant) or quite detailed (688 class submarine, Nimitz class aircraft carrier). The level of detail of the set of single-target classes dictates the complexity of the acoustic functions that define the uniqueness of a single-target class within the model.

Each single-target class is defined by a set of functions that describe its acoustic characteristics. These functions are like a library of acoustic signatures that an observer would compare to his observations to make classification decisions. In MUSAC II, however, the acoustic functions are in terms of average source levels instead of received signal levels, as would be the case with real signatures. There are three types of functions that describe a single-target class: narrowband, broadband, and modulated broadband functions.

The narrowband characteristics are described by a set of functions that define the average Lofar line level at specified frequencies. The narrowband functions have parameters of target speed and aspect angle. Different single-target classes may have different numbers of Lofar lines or different line levels. The total number of narrowband features in the model is determined by the total number of different line frequencies. If a target does not generate a Lofar line at a particular frequency in the set, then that narrowband function associated with the target is assigned zero power at that frequency.

The broadband characteristics are described by a set of functions that define the average source spectrum level of the target classes. These functions have parameters of frequency, target speed, and aspect angle. The broadband spectrum can be filtered into a set of broadband features. Thus rough classification clues can be derived from the presence or absence of low, medium, or high frequency broadband radiation. The broadband spectrum also acts as noise when detecting Lofar and Demon lines.

The modulated broadband characteristics are described by a set of functions that define the average modulation levels for a defined set of Demon lines. The parameters for these functions include broadband frequency, target speed, and aspect angle. To each Demon line, a modulation-level function is assigned; whereas, to each Lofar line, a line-level value is assigned. Thus the description of Demon features is more complicated than Lofar features.

Besides the user-defined single target classes there is a special target class, designated "nontarget," that will usually be included in the set of single-target classes. The nontarget is characterized by zero power for all the acoustic characteristics. This special target class is required to model the detection of the targets. If at a particular time step the observer has not detected one or more of the actual targets, then a hypothesis that contains the class of nontarget at the appropriate bearings will receive a high likelihood value.

B. Target Tracks

The second component in formulating multitarget hypotheses is a set of target tracks. A track is defined by the target's initial position and its course and speed as a function of time. From these functions the parameters of range, bearing, speed, and aspect angle of the target on a particular track can be calculated. It is important to note that the tracks are not limited to straight lines, and that only the target positions at the current time step need to be known. That is, if MUSAC II is formulated as an engagement model, then the decisions at one time step can affect the position of the observer and targets in later time steps.

Target tracks are thought of as having an existence independent of the target that is moving on them. The reason for this point of view is that hypothetical target configurations are constructed by assigning single-target classes to the tracks. The set of tracks will always include the tracks of the one true target configuration. In some applications of MUSAC II, hypothetical tracks may be defined in addition to the true tracks.

For most applications, only the tracks of the true targets will be defined. This assumes that the observer knows the range and bearing to the actual targets during the engagement. The primary purpose of defining the target tracks in MUSAC II is to provide the necessary kinematic parameters for use in the acoustic characteristic functions.

The acoustic functions and the track parameters are a representation of an acoustic signature library that the observer would compare to his observations for classification decision making. The perfect tracking assumption implies that the observer selects from the library only those signatures that correspond to the range, speed, and aspect of the true targets.

Hypothetical tracks can be defined for the purpose of modeling target position uncertainty. When additional tracks are defined, the MUSAC II methodology also performs a localization function in the sense that a hypothesis with target position close to the true target position will receive a higher likelihood value than a hypothesis that places targets away from the observed position. The addition of hypothetical tracks will result in more hypotheses. The most limiting feature of the MUSAC II methodology is the potential large size of the hypothesis set. For any particular application of the model, the analyst should try to minimize the number of hypotheses to save unnecessary calculations.

C. Multitarget Hypotheses

The complete multitarget hypothesis consists of an ordered arrangement of the possible single-target classes assigned to particular tracks. The formulation is usually done by combinatorial methods for combining the single-target classes with the target tracks. The hypothesis set is under the control of the analyst and should reflect the assumptions about the classification problem that is being studied.

To formulate the hypothesis set by the combinatorial method, all possible n-tuples of the single-target classes are enumerated. For this type of hypothesis structuring there will be k^n multitarget hypotheses generated, where k is the number of single target classes and n is the total number of tracks. For example, let (HVU, LVU, NON) be a set of single-target classes, where HVU is a high value unit, LVU is a low value unit, and NON is the special target class nontarget. Also assume that two tracks have been defined. Then the set of hypotheses is the set of 2-tuples, in which the first element is the target class assigned to track #1 and the second element of the target class assigned to track #2:

$$
\begin{array}{lll}
H_1 = \{\text{HVU, HVU}\} , & H_4 = \{\text{LVU, HVU}\} , & H_7 = \{\text{NON, HVU}\} \\
H_2 = \{\text{HVU, LVU}\} , & H_5 = \{\text{LVU, LVU}\} , & H_8 = \{\text{NON, LVU}\} \\
H_3 = \{\text{HVU, NON}\} , & H_6 = \{\text{LVU, NON}\} , & H_9 = \{\text{NON, NON}\}
\end{array}
$$

This hypothesis formulation might represent a scenario in which there are two real targets, an HVU on track #1, and an LVU on track #2. Thus the second hypothesis is the true hypothesis: $H_0 = H_2$. The observer expects to encounter one or two ships that are HVUs or LVUs. The single-target class, nontarget, is required to model the detection of one or both of the targets.

A hypothesis set that is generated by combinatorial means should be reviewed by the analyst, and unreasonable hypotheses be deleted to avoid unnecessary calculation. In forming the multitarget hypothesis the analyst should consider the maximum number of targets that the observer expects to encounter and the maximum number of occurrences of a particular single-target class within a hypothesis. In the previous example, if it is unlikely to encounter two HVUs, then the first hypothesis should be deleted. For each hypothesis there must be a single-target class assignment to each track. When the number of hypothesized targets is less than the number of defined tracks, then the special class of nontarget is assigned to the remaining tracks.

III SONAR DETECTION MODEL

The sonar detection model is based on a two-channel comparison of time-averaged power. The inputs to the model are source levels, noise levels, propagation loss, etc.; the input levels are allowed to vary rancomly in time to simulate signal fade and jump. The outputs of the model are the probabilities of detection of acoustic features such as Lofar lines. The model is a fairly simple representation of a sonar system. The MUSAC II user may replace this model with one of his own creation, so long as the outputs of the new model are also feature detection probabilities.

A. Input Parameters

1. Subscript Definitions

The subscripts i, j, k, m, n are used in the sonar model. As an aid to understanding the subscripted functions given later, the subscripts are first discussed:

i = Hypothesis identifier; i = 0 designates the true target configuration and i = 1,2,... designates the hypothetical target configurations.

j = Feature identifier; features are associated with frequency bands on sonar array/display combinations. As an example, features j = 1,2,...,20 can be associated with 20 possible Lofar lines from an omnidirectional array; features j = 21,...,40 can be associated with the same 20 possible Lofar lines from a towed array; features j = 41,42 can be associated with the low and high bands of the BTR from a spherical array; features j = 43,...,47 can be associated with five possible Demon lines in the 1-2 kHz band; features j = 48,...,52 can be associated with the same five Demon lines in the 2-4 kHz band; and features j = 53,...,57 can be associated with the same five

Demon lines in the 4-8 kHz band, where the multiband Demon information is from a spherical array. Aural classification from headphone information can be simulated by Lofar features and multiband Demon features.

k = Look angle identifier. The sonar associated with a feature is pointed in various directions. The maximum value of k is indexed on j so that a different number of look angles can be specified for different sonar arrays. For example, only one "look angle" is needed for an omnidirectional array, whereas many angles are needed for a preformed beam array.

m = Target track identifier. A set of tracks (time varying positions) is defined for the model. The real target configuration is constructed by assigning the real targets to their tracks. The hypothetical target configurations are constructed by assigning hypothetical targets to the tracks; nontargets (zero power) may be included in the hypothetical configurations also. The m-index identifies a track; however, a track may be assigned many different target types. The m-index, by itself, is not a target type identifier; but the combination (i,m) does identify a target type at a position in space and time.

n = Noise type identifier; different kinds of noise can be specified. For example, n = 1 can be sea state noise; n = 2 can be shipping noise; and n = 3 can be self noise.

2. Target Characteristics

The input functions that describe the single target class characteristics are the average values of the narrowband, broadband, and modulation levels.

$\overline{P}_{ijm}(v,\alpha)$ = Average narrowband source level (dB relative to 1 μPa^2 at 1 yd); mean squared pressure of the Lofar line associated with the jth feature one yard from the (i,m) target. The line level may be a function of target speed v and aspect angle α. The input to a computer program would not be indexed on i and m, but instead on a single target class ℓ. An additional vector, subscripted with "im" and composed of integer components ℓ, would designate the target class ℓ that is associated with the (i,m) target. Therefore, the computer input functions would be subscripted with "ℓj"; however, the equivalent subscripts "ijm" are used to describe the model.

$\overline{P}'_{im}(f,v,\alpha)$ = Average broadband source spectrum level (dB relative to 1 $\mu Pa^2/Hz$ at 1 yd); mean square pressure per unit frequency at one yard from the (i,m) target. The spectrum level is a function of frequency f, target speed v, and aspect angle α.

$\overline{M}_{ijm}(f,v,\alpha)$ = Average broadband modulation level (dB relative to 1.0); defined as the dB level of the square of the modulation index. The modulation index is the maximum amplitude minus the minimum amplitude divided by twice the average amplitude. The modulation level is a function of frequency, speed, and aspect; and it is indexed on the Demon line associated with the jth feature for the (i,m) target.

3. Track Input

The tracks are calculated from input values of initial positions and time-varying courses and speeds for the observer and targets. The target range r, relative bearing θ, and aspect angle α are then derived from the x,y positions of the units.

$x(t_o)$, $y(t_o)$ = Position of the observer at time $t = t_o$.

$x_m(t_o)$, $y_m(t_o)$ = Position of the mth target at time $t = t_o$.

$\beta(t)$, $u(t)$ = Course and speed (deg, kt) of the observer as a function of time.

$\varphi_m(t)$, $v_m(t)$ = Course and speed (deg, kt) of the mth target track as a function of time.

4. Environment and Sonar System Characteristics

The input parameters that characterize the environment are the average propagation loss and the array output average noise function. The sonar system is characterized by the array output average noise function, the beam pattern, the frequency response, the processor averaging time, the detection threshold, the data rate, and the sonar look angles.

$\overline{A}_i(f,r)$ = Average propagation loss (dB); mean squared pressure at one yard from the target divided by mean squared pressure at a range of r nmi from the target. Propagation loss is a function of frequency and range and is subscripted with i to denote that two propagation functions could be used: one function for the true conditions, and another function for the hypothesized propagation conditions.

$\overline{N}'_{0jn}(f,u,\lambda)$ = Average broadband, output noise spectrum level (dB relative to 1 $\mu Pa^2/Hz$); mean square pressure per unit frequency of the nth noise source at the output of the sonar array associated with the jth feature. It is a function of frequency f, observer speed u, and look angle λ. If the noise is isotropic, then $\overline{N}'$ is the noise outside the array minus the directivity index. In the nonisotropic case, the value of $\overline{N}'$ may be a function of look angle λ. The i = 0 index indicates that the true output noise spectrum level is used for both the real and hypothetical configurations.

$B_j(f,\lambda,\theta)$ = Beam pattern ratio ($0 \le B \le 1$); mean square voltage when the sonar is looking at angle λ and a single point-source target is on bearing θ, divided by the mean square voltage when the sonar is looking at the target. The beam pattern depends on the array associated with the jth feature, and the pattern is a function of frequency.

$B_j^*(f)$ = Side lobe ratio ($0 \le B^* \le 1$); the nominal value of $B_j(f,\lambda,\theta)$ when λ and θ are well separated.

$G_j(f)$ = Normalized frequency response ratio ($0 \le G \le 1$); output mean square pressure (voltage) divided by input mean square pressure for the frequency band associated with the jth feature. The function can include effects of hydrophone response, band filtering, or psychoacoustic frequency response.

f_j = Center frequency (Hz) associated with the jth feature (geometrical mean of lower and upper frequency limits).

W_j = Bandwidth (Hz) associated with the jth feature. If the frequency response is a unit rectangular function, then f_j and W_j completely define $G_j(f)$.

T_j = Averaging time (sec) of the signal processor associated with the jth feature.

d_j = Detection threshold ($d > 0$); the number of reference-channel standard deviations by which the data channel output must exceed the reference channel mean so that the jth feature is detected.

n_j = Number of independent observation opportunities on the jth feature during the computer time step ($1 \le n_j \le$ computer time step divided by the signal processor averaging time).

λ_{jk} = Value of the kth look angle (degrees from observer's heading) for the array associated with the jth feature. One model design would be to let the sonar look at only the target bearings: $\lambda_{jk} = \theta_m(t)$. An alternative model design would be to let the sonar look at bearings at equal increments; for example, 6° apart with $k = 1,\ldots,60$ to cover 360°.

5. Random Process Parameters

Five input functions for the real target configuration are calculated from a random process that is correlated in time. The input parameters that characterize the random process are the standard deviation, relaxation time, and mixing constant.

s_r = Standard deviation (dB) of the random process on the rth input function ($r = 1,2,3,4,5$).

τ_r = Relaxation time (min) of the rth random process.

c_r = Mixing constant ($0 \le c \le 1$) for the random process; c = 0 indicates a pure Gauss-Markov process; c = 1 indicates a pure lambda-sigma jump process.

B. Input Parameter Random Process

Real sonar signals fade in and out and make sudden jumps. A mixed random process that was originally proposed by Wagner* is used to model this phenomenon; the process is a combination of a Gauss-Markov and lambda-sigma jump random process. The Gauss-Markov process represents smooth changes in signal level, and the lambda-sigma jump process represents sudden changes in signal level.

1. General Equations

For each engagement run of the model, a different time history of an input parameter can be generated. The input parameter used at a given time step is calculated by drawing three independent random numbers x, y, z. With these random numbers, two zero-mean random values are computed and then mixed together:

$$J_n = J_{n-1}(1-x) + s\,x\,y$$

$$K_n = \rho K_{n-1} + s\sqrt{1-\rho^2}\,z$$

$$L_n = cJ_n + \sqrt{1-c^2}\,K_n$$

where

J_n = lambda-sigma jump random increment at nth time step.

K_n = Gauss-Markov random increment at nth time step.

L_n = mixed random increment at nth time step; L_n is added to the average value of the parameter to calculate value of the parameter at the nth time step.

---

*"A Comparison of Detection Models Used in ASW Operations Analysis (U)," D. H. Wagner Associates (October 1973).

c = mixing constant ($0 \le c \le 1$); $c = 0$ indicates a pure Gauss-Markov process, $c = 1$ indicates a pure lambda-sigma jump process.

s = standard deviation of the input parameter (dB).

ρ = $e^{-\Delta t/\tau}$; Gauss-Markov step-to-step correlation; and lambda-sigma jump probability that a jump did not occur during the last time step.

Δt = time step duration (min).

τ = Gauss-Markov relaxation time (min), and lambda-sigma jump mean time between jumps.

x = random number from a Bernoulli distribution with parameter $E(x) = 1 - \rho$. The $x = 0$ case means that no jumps occurred in the last time step; the $x = 1$ case means that at least one jump occurred in the last time step.

y,z = random numbers from a normal distribution with zero mean and unit variance.

The mixed random process, as defined above, has the statistics: $E(L_n) = 0$, $E(L_n^2) = s^2$, and $E(L_n L_{n-1}) = s^2\rho$.

2. Generation of Input Functions

The equation that generates one of the five input functions is given below. The other four equations are omitted because they are similar and add nothing new to the description of the model. The randomization is applied only to the input functions for the true target configuration because the hypothetical signals must be based on engagement-to-engagement average values, not on detailed knowledge of a particular engagement. The randomly generated input function is denoted by the same symbol that is used in the input list, but omitting the average sign. Notice that the generated input function is defined in units of power ratios, not dB.

$$P_{ijm}(v,\alpha) = \begin{cases} 10^{\,.1[\overline{P}_{0jm}(v,\alpha) + \Delta P_{0jm}]} & \text{for} \quad i = 0 \\ 10^{\,.1[\overline{P}_{ijm}(v,\alpha) + \delta_1]} & \text{for} \quad i = 1,2,3,\ldots \end{cases}$$

where

$\overline{P}_{ijm}(v,\alpha)$ = Average narrowband source level (dB)

ΔP_{0jm} = Random increment (dB); computed from the mixed random process using the parameters s_1, τ_1, and c_1. ΔP is the same as L_n in the previous section.

δ_1 = Increment (dB) that must be added to the mean level (dB) so that the mean power (ratio) is correctly converted. The input power is assumed to be lognormally distributed, therefore the increment is:

$$\delta_1 = s_1^2/8.68 \quad ,$$

where s_1 is the standard deviation (dB) of the random process on the narrowband source level.

A new random increment is computed at each time step (not explicitly denoted), for each feature (j), and for each target (o,m). Thus, if there are 10 time steps, 5 features, and 2 targets, then 100 random increments will be calculated to generate the input values of narrowband source power. The value of random increment is assumed to be independent of the underlying values of speed, aspect, frequency, etc.

The generation of input functions from the five mixed random processes produces the following functions:

$P_{ijm}(v,\alpha)$ = Narrowband source power (μPa^2 at 1 yd)

$P'_{im}(f,v,\alpha)$ = Broadband source spectrum ($\mu Pa^2/Hz$ at 1 yd)

$M_{ijm}(f,v,\alpha)$ = Broadband modulation ratio ($0 \leq M \leq 1$)

$A_i(f,r)$ = Propagation loss ratio ($A \geq 1$) from one yard

$N'_{0jn}(f,u,\lambda)$ = Broadband, array output, noise spectrum ($\mu Pa^2/Hz$).

C. Array Output Signal and Noise

The power at the output of the sonar array is calculated for the narrowband, broadband, and modulated broadband cases.

1. Narrowband

The target signal that is outside the array is calculated by reducing the source power by the propagation loss:

$$Q_{ijm} = \frac{P_{ijm}(v_m, \alpha_m)}{A_i(f_j, r_m)} \quad .$$

The mean square pressure from all targets at the output of the array is then calculated by reducing the signal with the beam pattern ratio and summing over all target sources:

$$S_{ijk} = \sum_m Q_{ijm} \; B_j(f_j, \lambda_{jk}, \theta_m) \quad .$$

In addition to the narrowband signal, the following function is needed in later calculations:

$$V_{ijk} = \sum_m [Q_{ijm} \; B_j(f_j, \lambda_{jk}, \theta_m)]^2 \quad .$$

2. Broadband

The target spectrum that is outside the array is calculated by reducing the source spectrum by the propagation loss:

$$Q'_{im}(f) = \frac{P'_{im}(f, v_m, \alpha_m)}{A_i(f, r_m)} \quad .$$

The mean square pressure per unit frequency due to all targets at the output of the array is then calculated by reducing the individual signals with the beam pattern ratio, summing over all target sources, and multiplying by the frequency response:

$$S'_{ijk}(f) = G_j(f) \sum_m Q'_{im}(f) \; B_j(f, \lambda_{jk}, \theta_m) \quad .$$

The side lobe interference spectrum is also needed:

$$S_{O\,j}'^{*}(f) \;=\; G_j(f)\; B_j^{*}(f) \sum_m Q_{Om}'(f) \quad .$$

Finally, the noise spectrum at the output of the array is the sum from all noise sources:

$$N_{O\,jk}'(f) \;=\; G_j(f) \sum_n N_{O\,jn}'(f,u,\lambda_{jk}) \quad .$$

3. Modulated Broadband

The additional increment of broadband signal power (mean squared pressure per unit frequency) due to the modulation is calculated by multiplying the modulation function times the output broadband signal, summing over all targets, and multiplying by one half the frequency response:

$$\Delta S_{i\,jk}'(f) \;=\; \frac{1}{2}\, G_j(f) \sum_m Q_{im}'(f)\; B_j(f,\lambda_{jk},\theta_m)\; M_{ijm}(f,v_m,\alpha_m) \quad .$$

D. Signal Processor Statistics

The mean and variance of the output of the signal processor are required to calculate the probability of detection. The sonar model assumes that the signal processor has two channels: (1) a data channel that produces a random value from a normal distribution of mean μ and variance σ^2; and (2) a reference channel that produces two deterministic parameters μ^* and σ^{*2} that are measures of the background noise in which the data signal is to be detected. The equations for the signal processor statistics are not obvious. In the interest of a short description of the sonar model, the derivation of the equations is deferred to Appendix A.

1. Narrowband

The broadband power in a narrowband of width W_j is:

$$R_{ijk} \;=\; [S_{i\,jk}'(f_j) + N_{Ojk}'(f_j)]W_j \quad .$$

With this definition, the channel statistics for the narrowband case are written:

Data Channel

$$\mu_{ijk} = S_{ijk} + R_{ijk} \qquad \text{for } i = 0,1,2,\ldots$$

$$\sigma^2_{ijk} = S^2_{ijk} - V_{ijk} + \frac{2}{W_j T_j} S_{ijk} R_{ijk} + \frac{1}{W_j T_j} R^2_{ijk}$$

Reference Channel

$$\mu^*_{0jk} = R_{0jk}$$

$$\sigma^{*2}_{0jk} = \frac{1}{W_j T_j} R^2_{0jk}$$

2. Broadband

The broadband power is calculated by integrating over frequency. Since the frequency response function $G_j(f)$ is defined as including the band cutoffs, the integration is theoretically from zero to infinity. The signal and noise spectra are combined into a data spectrum and a background spectrum:

$$R'_{ijk}(f) = S'_{ijk}(f) + N'_{0jk}(f)$$

$$R'^*_{0jk}(f) = S'^*_{0j}(f) + N'_{0jk}(f) \quad .$$

With these two definitions the channel statistics for the broadband case are written:

Data Channel

$$\mu_{ijk} = \int R'_{ijk}(f)\, df$$

$$\sigma^2_{ijk} = \frac{1}{T_j} \int [R'_{ijk}(f)]^2\, df \qquad \text{for} \quad i = 0,1,2,\ldots$$

Reference Channel

$$\mu^*_{0jk} = \int R'^*_{0jk}(f)\,df$$

$$\sigma^{*2}_{0jk} = \frac{1}{T_j}\int [R'^*_{0jk}(f)]^2\,df$$

3. Modulated Broadband

The squared average of a modulated broadband signal contains more power than the signal without modulation. The statistics for detection of modulated features uses this idea:

Data Channel

$$\mu_{ijk} = \int [\Delta S'_{ijk}(f) + R'_{ijk}(f)]\,df$$

for $i = 0,1,2,\ldots$

$$\sigma^2_{ijk} = \frac{1}{T_j}\int [\Delta S'_{ijk}(f) + R'_{ijk}(f)]^2\,df$$

Reference Channel

$$\mu^*_{0jk} = \int R'_{0jk}(f)\,df$$

$$\sigma^{*2}_{0jk} = \frac{1}{T_j}\int [R'_{0jk}(f)]^2\,df \quad .$$

E. Feature Detection

1. Observed Data

The output of the data channel is compared to a threshold value a_{jk} that is a function of the reference channel parameters:

$$a_{jk} = \mu^*_{0jk} + d_j\,\sigma^*_{0jk} \quad .$$

If the data channel output is larger than the threshold, then the feature is assigned a value one; if the output is less than the threshold, then the feature is assigned a value zero.

The term "feature" is used here to mean feature "j" (for example a Lofar line) on look angle "k" produced by the real target configuration ($i = 0$). Features, whether they are of value one or zero, are observed data; they are used to classify the target configuration.

The data channel output is assumed to be normally distributed with mean μ_{0jk} and variance σ^2_{0jk}. A random number z_{jk} is drawn from a normal distribution with zero mean and unit variance. The random output of the data channel can be written:

$$y_{jk} = \mu_{0jk} + \sigma_{0jk} z_{jk} .$$

The Bernoulli distributed random feature x_{jk} is then determined by:

$$x_{jk} = \begin{cases} 1 & \text{if} \quad y_{jk} \geq a_{jk} \qquad \text{(detection)} \\ 0 & \text{if} \quad y_{jk} < a_{jk} \qquad \text{(no detection)} \end{cases} .$$

The time required to produce a feature is the processor averaging time T_j. Thus a new feature, of value zero or one, can be produced at a maximum rate of once every T_j seconds. Due to sonar system design, the rate may be less than the maximum. The input parameter that controls the rate is n_j: the number of times the feature x_{jk} is produced in one computer time step.

2. Hypothesized Feature Detection Probabilities

Feature detection probabilities are calculated for each hypothetical target configuration. The probability of detection is the probability that the normally distributed output of the hypothetical data channel is greater than the detection threshold:

$$p_{ijk} = \frac{1}{\sqrt{2\pi}\,\sigma_{ijk}} \int_{a_{jk}}^{\infty} e^{-\frac{1}{2}\left(\frac{y - \mu_{ijk}}{\sigma_{ijk}}\right)^2} dy .$$

The feature detection probability is the probability that the jth feature on the kth look angle would be detected if the ith hypothesis were true. These hypothesized probabilities of detection are used in conjunction with the observed data to calculate the likelihood of the data under each hypothetical configuration.

IV HYPOTHESIS PROBABILITY CALCULATION

The multitarget hypotheses are assigned probabilities called posteriors. The posterior probability is the probability that a particular hypothesis is true, given the observed data. The posteriors are random variables because they are computed from random observed data: the random x_{jk}'s derived in the previous chapter. To calculate the posteriors, the likelihoods are first computed and then Bayes' rule is applied.

A. Likelihood Calculation

The likelihood is the probability that all the data collected through the nth time step would occur if the ith hypothesis were true: $\mathrm{Prob}[D_n|H_i]$. Before the likelihood equations can be derived the x and p symbols of the sonar model must be altered somewhat to include a time step index n and an observation index ℓ:

$x_{jkn\ell}$ = Bernoulli random variable; x = 0 means no detection, and x = 1 means detection of the jth feature on the kth look angle for the ℓth observation during the nth time step.

p_{ijkn} = Probability that the jth feature on the kth look angle would be detected during the nth time step if the ith hypothesis were true. The value of p is the same for all observations during a given time step.

A new random variable ξ_{jkn} is defined as the number of detections of the jth feature on the kth look angle during the nth time step:

$$\xi_{jkn} = \sum_{\ell} x_{jkn\ell} \qquad \ell = 1,2,\ldots,n_j$$

The x-variables are assumed to be independent Bernoulli random variables, and therefore the ξ-variable is binomially distributed with parameters p_{ijkn} and n_j. The probability mass function of a binomial distribution is:

$$L_{ijkn} = \binom{n_j}{\xi_{jkn}} (p_{ijkn})^{\xi_{jkn}} (1 - p_{ijkn})^{n_j - \xi_{jkn}}$$

where

L_{ijkn} = Probability of exactly ξ_{jkn} detections in n_j observations. The equation can also be interpreted as the likelihood (probability) that the observed pattern of detections (of the jth feature on the kth look angle) would occur during the nth time step if the ith hypothesis were true.

$\binom{n}{\xi}$ = $\frac{n!}{\xi!(n-\xi)!}$ binomial coefficient

n_j = Number of observations of the jth feature during a time step.

ξ_{jkn} = Number of detections (of the jth feature on the kth look angle) during the nth time step.

The ξ-variables are assumed to be independent random variables over the index set (j,k,n). Therefore, the joint probability is a product of the individual probabilities:

$$\text{Prob}[D_n | H_i] = \prod_m \prod_k \prod_j L_{ijkm}$$

where

$\text{Prob}[D_n | H_i]$ = Likelihood that observed pattern of detections (of all features on all look angles over all memorable time steps, including the nth step) would occur if the ith hypothesis were true.

$j = 1,2,\ldots,j\text{max}$

$k = 1,2,\ldots,k\text{max}_j$

$m = n, n-1, n-2, \ldots, n-r+1$

r = Number of time steps for which patterns can be remembered; r is the length of memory of the classification process.

The ratio of two likelihoods is a measure of the diagonistic impact of the data on one hypothesis relative to another hypothesis. For example, if the likelihood ratio $L_1/L_2 = 10$, then the data is ten times more favorable to hypothesis H_1 than it is to H_2.

B. Posterior Calculation

Once the likelihoods are calculated, the posteriors are easily determined from Bayes' rule:

$$\text{Prob}[H_i \mid D_n] = \frac{\text{Prob}[D_n \mid H_i]\ \text{Prob}[H_i]}{\sum_i \text{Prob}[D_n \mid H_i]\ \text{Prob}[H_i]}$$

where

$\text{Prob}[H_i]$ = The a priori probability that the ith hypothesis is true. The priors are input constants that quantify intelligence data on the targets before any sonar data is gathered.

$\text{Prob}[H_i \mid D_n]$ = The posterior probability that the ith hypothesis is true after observing memorable sonar data through the nth time step.

The posterior probabilities are random, at-the-moment, probability estimates that are assigned to the hypotheses. The posteriors are not themselves decision probabilities, although the meaning of the phrase "probability of classification" could be defined as the posterior probability. Decision probabilities, such as probability of classification, are defined in the next section on decision making.

V DECISION MAKING

The decision making element of the MUSAC II methodology represents that portion of the classification process that determines the target classes and directs tactical action. A Bayesian decision criterion that uses the posterior probabilities and conditional values of decision outcomes is the basis of the decision making model.

A. Decisions and Values

The analyst must define a set of possible decisions, $B_1,\ldots,B_k$. Next a set of values $Val[B_k|H_i]$, of the kth decision, conditioned on the ith hypothesis being true, must be defined. The values are relative measures, in that they may represent the conditional monetary value, economic value, or utility value of the decision outcome. The exact formulation of the decision set and the value functions will depend on the particular application of the MUSAC II methodology. There are two general categories of decisions: tactical decisions and classification decisions.

1. Tactical Decisions

At some point in the engagement the observer must make tactical decisions based on his estimate of the composition of the target group. An example is the decision to launch a weapon at a particular target. Consider the example described in Chapter II where the hypothesis set consists of three possible target types HVU, LVU, and NON and two possible tracks. Assuming the objective of the observer is to attack the HVU, a possible decision set for tactical action is:

B_1: attack the target on track #1

B_2: attack the target on track #2

B_3: disengage and search for other targets.

Next a set of conditional values of the decision, assuming the ith hypothesis is true, must be defined. For example, the value for action B_1 for hypotheses that identify the HVU as being on track #1 would be greater than the value for the hypotheses that identify the LVU or non-target as being on track #1. The values for decision B_3 (disengage) would reflect missed opportunity values (costs) for hypotheses that identify the HVU being present. In general, the value set must reflect the objectives of the observer, and the internal consistency of the value set is of importance and not their absolute value. Although the values are somewhat loosely defined, the sensitivity of the decisions to the value structure can easily be determined. For a given set of replications of the MUSAC II model, the values of the posteriors can be saved and then different value structures can be tested with little additional computational effort.

2. Classification Decisions

The classification decisions can vary from determining the presence of one or more targets (detection) to completely describing the target group (designate a single hypothesis as true). As with the tactical decisions, the observer must define his classification decision set. Depending on the context of the problem at hand, the decision set can vary from a very gross description of the target group to a very detailed description. In the previous example, a possible decision set is:

B_1: HVU on track #1

B_2: HVU on track #2

B_3: HVU not present.

If it is assumed that the observer thinks that each decision has equal importance, then a simple value structure can be defined:

$$\mathrm{Val}[B_1|H_i] = \begin{cases} 1 & \text{if } H_i \text{ identifies HVU on track \#1} \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{Val}[B_2|H_i] = \begin{cases} 1 & \text{if } H_i \text{ identifies HVU on track \#2} \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{Val}[B_3|H_i] = \begin{cases} 1 & \text{if } H_i \text{ does not contain an HVU} \\ 0 & \text{otherwise} \end{cases} .$$

B. Bayes Decision Criterion

Bayes decision criterion is a rule that defines the best decision by calculating the average (expected) value of each decision given the observed data:

$$\mathrm{EVal}[B_k|D_n] = \sum_i \mathrm{Val}[B_k|H_i]\ \mathrm{Prob}[H_i|D_n] \quad ,$$

where $\mathrm{Prob}[H_i|D_n]$ is the posterior probability of the ith hypothesis, given the data D_n through the nth time step. The Bayes criterion selects the decision that has the maximum expected value:

$$\text{select decision } B^* \text{ such that}$$
$$\mathrm{EVal}[B^*|D_n] = \max_k\{\mathrm{EVal}[B_k|D_n]\} \quad .$$

In the classification decision example, the expected value of each decision is simply the summation of posterior probabilities for hypotheses related to the decision. The classification decision B^* is then the decision with the highest aggregated probability.

C. Deferred Decision Making

The decision maker may have the option to make the decision B^* as indicated by the Bayes criterion, or to defer the decision and collect additional information. To model this process, a new function is defined. The expected value of perfect information given the data, $\mathrm{EVal}[PI|D_n]$, is defined as the expected value of the best decision (sum of the values of the best decision for each hypothesis times the posterior probability of the hypothesis) minus the expected value of the Bayes decision:

$$\mathrm{EVal}[PI|D_n] = \sum_i \max_k\{\mathrm{Val}[B_k|H_i]\}\ \mathrm{Prob}[H_i|D_n] - \mathrm{EVal}[B^*|D_n] \quad .$$

In the limit as the posterior probability of a particular hypothesis approaches unity, the expected value of perfect information approaches zero.

The decision maker selects decision B^* at the first time step that the expected value of perfect information drops below an input threshold V_0:

if $\quad \mathrm{EVal}[PI|D_n] \le V_0 \quad ,$

then $\quad$ select decision $B^* \quad .$

If, however, the expected value of perfect information is greater than the threshold, then the decision maker will defer making a decision and will collect more information.

The idea behind this procedure is that if the expected value of the best decision is considerably higher than the current Bayes decision, then it pays to continue gathering information to make a better decision in the future. On the other hand, if the expected value of the best decision is only a little higher than the current Bayes decision, then it is better to select decision B^* because the gain, in expected value $EVal[PI|D_n]$, is not worth the risk V_0. For example, if a submariner thought he were closing an ASW capable target group, V_0 would reflect the risk associated with counterdetection and might be a relatively large value. Conversely, if the submariner did not expect a high ASW threat, V_0 would be a relatively low value. In the modeling sense V_0 is a control parameter--high values cause quick decisions with higher chances of selecting the wrong decision, and low values delay the decision until very conclusive acoustic information is incorporated in the posterior probabilities. V_0 need not be a constant over the engagement. For example, in the initial stages of the engagement V_0 might be a low value because there is little risk to the observer because the targets are at long range. As the observer closes the targets, the risks of counter-detection increase and the value of V_0 should be increased accordingly.

In the classification decision example, the threshold parameter V_0 can be interpreted as a probability threshold, or a measure of acceptable uncertainty. If V_0 is a low value, say 0.05, then the expected value of a decision must exceed 0.95 for the decision to be made, otherwise it will be deferred.

D. Decision Probabilities

From Monte Carlo replications of the MUSAC II model the probability of making each decision can be estimated. To calculate the probability of making a specified decision at a particular time step, a new random variable is defined:

$$x_{knr} = \begin{cases} 1 & \text{if} \quad B_k = B_r^* \text{ at } t = t_{nr} \\ 0 & \text{otherwise} \end{cases} .$$

That is, if the kth decision is made at the nth time step on the rth replication, x_{knr} is set to 1, and is 0 otherwise. The probability of making the kth decision at the nth time step can then be estimated by:

$$\text{Prob}[B_k D_n] = \frac{1}{R} \sum_{r=1}^{R} x_{knr}$$

where R is the total number of replications. These probabilities can be summed over the time steps to estimate the probability of making decision B_k at any time in the engagement; or the $\text{Prob}[B_k D_n]$ can be summed over the decisions to estimate the probability of making some decision at a particular time step. The probability of some decision at some time is the double sum over time and decisions.

Correct and incorrect classification probabilities are examples of decision probabilities. In the classification decision example, if the HVU were truly on track #1, then the probability of correct classification is:

$$\text{Prob}[CC] = \sum_n \text{Prob}[B_1 D_n] .$$

The probability of incorrect classification is:

$$\text{Prob}[IC] = \sum_n \left(\text{Prob}[B_2 D_n] + \text{Prob}[B_3 D_n]\right) ,$$

and the probability of no classification is:

$$\text{Prob}[NC] = 1 - \sum_n \sum_k \text{Prob}[B_k D_n] .$$

E. Engagement MOE

The expected value of the engagement can be used as an overall measure of effectiveness (MOE) to study the effects of parameter variations. To compute the MOE, the actual value of the decision is recorded and averaged over the replications:

$$\text{MOE} = \frac{1}{R}\sum_{r} \text{Val}[B_r^*|H_0] \quad ,$$

where B_r^* is the decision on the rth replication, H_0 is the true hypothesis, and R is the total number of replications. In the classification decision example, the engagement MOE is identical to the probability of correct classification.

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Appendix A

DERIVATION OF SIGNAL PROCESSOR STATISTICS

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DERIVATION OF SIGNAL PROCESSOR STATISTICS

A rationale for the channel output statistics is presented in this appendix. The first section derives general equations for the statistics of the squared magnitude of a sum of independent random vectors. These equations are used in various ways in the next three sections; the sections present the assumptions and derivations for the narrowband, broadband, and modulated broadband equations.

A. Statistics of the Squared Magnitude of a Sum of Independent Random Vectors

The random variable P is defined as:

$$P = \left|\sum_i \vec{A}_i\right|^2$$

where the $\vec{A}_i$'s are vectors. If the magnitude of $\vec{A}_i$ is the rms pressure from the ith source, then P represents the power from all sources at a given frequency.

The above equation can be rewritten as a double sum of dot products between the vectors:

$$P = \left(\sum_i \vec{A}_i\right) \cdot \left(\sum_j \vec{A}_j\right)$$

$$P = \sum_i \sum_j A_i \ A_j \ \cos(\theta_i - \theta_j)$$

where A_i is the magnitude and θ_i is the phase angle of the ith vector.

Two assumptions are made: the A_i's and θ_i's are all independent random variables; and the θ_i's are uniformly distributed from 0 to 2π radians. Under the independence assumptions, the expected value of P is:

$$E(P) = \sum_i E(A_i^2) + \sum\sum_{i \neq j} E(A_i)\, E(A_j)\, E(\cos\, \theta_{ij}) \quad ,$$

where $\theta_{ij} = \theta_i - \theta_j$. Under the uniform angle assumption, the expected values of the cosine terms are zero. Therefore, the expected value of P is:

$$E(P) = \sum_i E(A_i^2) \quad .$$

In other words, the average total power from many independent sources is the sum of average power from each source.

The expected value of the square of P is:

$$E(P^2) = \sum_i \sum_j \sum_m \sum_n E(A_i A_j A_m A_n \cos\, \theta_{ij}\, \cos\, \theta_{mn}) \quad .$$

When none of the indices are equal, the expected value of the argument is zero because the expected values of both cosine terms are zero. Likewise, when any three of the indices are equal but the fourth index is not, the expected value of the argument is zero because the expected value of one of the cosine terms is zero (the other cosine term has value one). When all indices are equal, the expected value of the argument is:

$$E(A_i^4) \quad .$$

When $i = j$ and $m = n$ and $i \neq m$, the expected value of the argument is:

$$E(A_i^2)\, E(A_m^2) \quad .$$

And finally, when $i = m$ and $j = n$ and $i \neq j$, and when $i = n$ and $j = m$ and $i \neq j$, the expected values of the argument are:

$$E(A_i^2)\, E(A_j^2)\, E(\cos^2\, \theta_{ij})$$

and

$$E(A_i^2)\, E(A_j^2)\, E(\cos\, \theta_{ij}\, \cos\, \theta_{ji}) \quad .$$

The expected value of the square of the cosine is 1/2.

Combining the above results, the expected value of P^2 is:

$$E(P^2) = \sum_i E(A_i^4) + 2 \sum\sum_{i \neq j} E(A_i^2)\ E(A_j^2) \quad .$$

The square of the expected value of P is:

$$E^2(P) = \left[\sum_i E(A_i^2)\right]^2 = \sum_i \sum_j E(A_i^2)\ E(A_j^2)$$

$$E^2(P) = \sum_i E^2(A_i^2) + \sum\sum_{i \neq j} E(A_i^2)\ E(A_j^2) \quad .$$

And therefore the variance of P is:

$$\mathrm{Var}(P) = E(P^2) - E^2(P)$$

$$\mathrm{Var}(P) = \sum_i \mathrm{Var}(A_i^2) + \sum\sum_{i \neq j} E(A_i^2)\ E(A_j^2) \quad .$$

In other words, the variance of the total power from many independent sources is <u>larger</u> than the sum of the variances of the power from each source.

The next task is to calculate the covariance between two squared magnitudes of sums of independent random vectors:

$$P = \sum_i \sum_j A_i\ A_j\ \cos(\theta_i - \theta_j)$$

$$Q = \sum_m \sum_n B_m\ B_n\ \cos(\varphi_m - \varphi_n)$$

where A_i θ_i B_m φ_m are all independent random variables except that A_i is not independent of B_i, and θ_i is not independent of φ_i (however, A_i is independent of θ_i, etc.). The angles θ_i and φ_m are assumed to be uniformly distributed from 0 to 2π radians.

The expected value of the product is:

$$E(PQ) = \sum_i \sum_j \sum_m \sum_n E(A_i\ A_j\ B_m\ B_n\ \cos\theta_{ij}\ \cos\varphi_{mn})$$

where $\theta_{ij} = \theta_i - \theta_j$ and $\varphi_{mn} = \varphi_m - \varphi_n$. When none of the indices are equal, or when any three indices are equal but the fourth is not equal, the expected value of the argument is zero. When all indices are equal, the expected value of the argument is:

$$E(A_i^2 B_i^2) \quad .$$

When $i = j$ and $m = n$ and $i \neq m$, the expected value of the argument is:

$$E(A_i^2)\ E(B_m^2) \quad .$$

When $i = m$ and $j = n$ and $i \neq j$, and when $i = n$ and $j = m$ and $i \neq j$, the expected values of the argument are:

$$E(A_i B_i)\ E(A_j B_j)\ E(\cos\theta_{ij}\ \cos\varphi_{ij})$$

$$E(A_i B_i)\ E(A_j B_j)\ E(\cos\theta_{ji}\ \cos\varphi_{ji})$$

Combining the above results, the expected value of PQ is:

$$E(PQ) = \sum_i E(A_i^2 B_i^2) + \sum_{i\neq j}\sum E(A_i^2)\ E(B_j^2)$$
$$+\ 2\sum_{i\neq j}\sum E(A_i B_i)\ E(A_j B_j)\ E(\cos\theta_{ij}\ \cos\varphi_{ij}) \quad .$$

The product of expected values is:

$$E(P)\ E(Q) = \sum_i E(A_i^2) \sum_j E(B_j^2)$$

$$E(P)\ E(Q) = \sum_i E(A_i^2)\ E(B_i^2) + \sum_{i\neq j}\sum E(A_i^2)\ E(B_j^2) \quad .$$

Therefore, the covariance between P and Q is:

$$\mathrm{Cov}(P,Q) = E(PQ) - E(P)\ E(Q)$$

$$\mathrm{Cov}(P,Q) = \sum_i \mathrm{Cov}(A_i^2, B_i^2)$$
$$+\ 2\sum_{i\neq j}\sum E(A_i B_i)\ E(A_j B_j)\ E(\cos\theta_{ij}\ \cos\varphi_{ij}) \quad .$$

Note that $\mathrm{Cov}(P,Q)$ reduces to the previously derived $\mathrm{Var}(P)$ when $A_i = B_i$ and $\theta_i = \varphi_i$.

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B. Narrowband Statistics

The equations for μ and σ^2 for the narrowband case are derived by assuming that the amplitude A_{in} and phase θ_{in} from the ith source in the nth frequency bin have the statistics as defined in Table 1. The symbols A_{in} A_{im} θ_{in} θ_{im} are being used in the same sense as the symbols A_i B_i θ_i φ_i in the previous section.

The assumptions are discussed below by equation number:

(1) The signal power and the noise power are assumed to be equally distributed over the narrowband of width W.

(2) The variance of the signal is zero; this means that the Lofar line amplitude is assumed to be constant. The variance of the noise power is the square of the mean noise power, this is a Gaussian noise assumption.

(3) The bin-to-bin covariance of the signal power is zero because the amplitude is constant. The covariance of the noise power is zero because the noise is assumed to be independent from bin-to-bin.

(4) The expected value of the signal amplitude product is equal to the expected signal power because the signal amplitude is constant. By assuming that the bin-to-bin covariance of the noise amplitude is zero, the expected value of the product of noise amplitude is equal to the square of the expected value of the noise amplitude. The noise amplitude A is Rayleigh distributed with mean $(\pi E(A^2)/4)^{\frac{1}{2}}$.

(5) The bin-to-bin covariance of the product of the signal cosines is 1/2 because the phase angle from a given source is assumed to be completely correlated from bin to bin. The covariance is zero for noise phase because it is assumed to be independent from bin to bin. The covariances involving signal phase with noise phase are also zero because they are assumed independent.

The mean power from all sources over all frequency bins is:

$$E(P) = \sum_n \sum_i E(A_{in}^2) \qquad \begin{array}{l} i = 1,2,\ldots,s+r \\ n = 1,2,\ldots,WT \end{array}$$

$$E(P) = WT\left[\sum \frac{S_i}{WT} + \sum \frac{R_k}{WT}\right]$$

Table 1

SIGNAL AND NOISE STATISTICS

		Signal $i = 1,2,\ldots,s$	Noise $i = s+1,\ldots,s+r$; $k = i-s$		
(1)	$E(A_{in}^2) =$	$\frac{S_i}{WT}$	$\frac{R_k}{WT}$		
(2)	$Var(A_{in}^2) =$	0	$\left(\frac{R_k}{WT}\right)^2$		
(3)	$Cov(A_{in}^2, A_{im}^2) =$	0	0		$m \neq n$
(4)	$E(A_{in}A_{im}) =$	$\frac{S_i}{WT}$	$\frac{\pi}{4}\frac{R_k}{WT}$		$m \neq n$
(5)	$E(\cos\theta_{ijn}\cos\theta_{ijm}) =$	$\frac{1}{2}$	0	for $j = 1,\ldots,s$	$m \neq n$
		0	0	for $j = s+1,\ldots,s+r$	

Note: $\theta_{ijn} = \theta_{in} - \theta_{jn}$; the difference of the ith and jth phase angles.

WT = Number of frequency bins; bandwidth times averaging time ($n = 1,2,\ldots, WT$).

therefore, $$\mu = S + R$$

where $$S = \sum S_i \qquad i = 1,2,\ldots,s$$

$$R = \sum R_k \qquad k = 1,2,\ldots,r$$

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The variance of the power is computed from the formulae derived in the previous section. The variance of the power in the nth bin is:

$$\mathrm{Var}(P_n) = \sum_i \mathrm{Var}(A_{in}^2) + \sum_{i\neq j}\sum E(A_{in}^2)E(A_{jn}^2)$$

$$\mathrm{Var}(P_n) = \sum \left(\frac{R_k}{WT}\right)^2 + \left[\sum \frac{S_i}{WT} + \sum \frac{R_k}{WT}\right]^2 - \left[\sum \left(\frac{S_i}{WT}\right)^2 + \sum \left(\frac{R_k}{WT}\right)^2\right]$$

$$\mathrm{Var}(P_n) = \left(\frac{1}{WT}\right)^2 \left[(S+R)^2 - V\right] \quad ,$$

where $$V = \sum S_i^2 \qquad i = 1,\ldots,s \quad .$$

The covariance of the power in the nth bin with the power in the mth bin is:

$$\mathrm{Cov}(P_n\,,\,P_m) = \sum_i \mathrm{Cov}(A_{in}^2, A_{im}^2)$$

$$+ 2\sum_{i\neq j}\sum E(A_{in}A_{im})\,E(A_{jn}A_{jm})\,E(\cos\theta_{ijn}\ \cos\theta_{ijm})$$

$$\mathrm{Cov}(P_n\,,\,P_m) = \sum_{i\neq j}\sum E(A_{in}A_{im})\,E(A_{jn}A_{jm}) \qquad \begin{array}{l} i = 1,\ldots,s \\ j = 1,\ldots,s \end{array}$$

$$\mathrm{Cov}(P_n\,,\,P_m) = \left[\sum \frac{S_i}{WT}\right]^2 - \sum \left(\frac{S_i}{WT}\right)^2$$

$$\mathrm{Cov}(P_n\,,\,P_m) = \left(\frac{1}{WT}\right)^2 \left[S^2 - V\right] \quad .$$

The variance of the power from all bins is:

$$\mathrm{Var}(P) = \sum_n \mathrm{Var}(P_n) + \sum_{n\neq m}\sum \mathrm{Cov}(P_n\,,\,P_m) \qquad \begin{array}{l} n = 1,\ldots,WT \\ m = 1,\ldots,WT \end{array}$$

$$\mathrm{Var}(P) = \frac{1}{WT}\left[(S+R)^2 - V\right] + \left(1 - \frac{1}{WT}\right)\left[S^2 - V\right]$$

therefore,

$$\sigma^2 = S^2 - V + \frac{2}{WT}SR + \frac{1}{WT}R^2 \quad .$$

As an example, assume just one signal of power S_1 and no noise (R = 0), then the variance is:

$$\sigma^2 = S_1^2 - S_1^2 = 0 \quad ,$$

as it should be since the amplitude of the signal is assumed to be constant. If, however, there are two signals of power S_1 and S_2 and no noise, then the variance is:

$$\sigma^2 = (S_1 + S_2)^2 - (S_1^2 + S_2^2) = 2\ S_1 S_2 \quad .$$

In this case the two signals have interfered randomly with each other because of their random values of phase angle. If $S_1 = A_1^2$ and $S_2 = A_2^2$, then the minimum power is $(A_1 - A_2)^2$, the maximum power is $(A_1 + A_2)^2$, and the variance due to random values inbetween is $2\ A_1^2 A_2^2$.

The distribution of power caused by random interference is not normally distributed. If, however, there are many independent signals, then the total power is approximately normally distributed (usually) with mean μ and variance σ^2 as derived above. The normal distribution assumption is not very good when there are only a few sources, as will be the case with a MUSAC II application. There is, however, a compensating effect that will tend to reduce the errors involved by assuming a normal distribution. The most critical time for having an accurate detection model occurs when the lines are just being detected. In this case the signals are all small and the variance is approximately R^2/WT. A normal distribution for the small signal case is reasonable because the power is averaged over WT bins.

In addition to the question of the shape of the distribution, there is also the question of observation-to-observation independence. It is reasonable to assume that the noise is independent from one observation to another, but the source signals may not be independent. This is because the random phase angle may not change fast enough compared to the averaging time. The argument for assuming independence uses the same point as raised in assuming a normal distribution: for the cases of interest, the Lofar signal will be small; thus the noise terms in the variance will dominate, and therefore the observation-to-observation independence of noise will also dominate.

The mean and variance of the data channel were derived as:

$$\mu = S + R$$

$$\sigma^2 = S^2 - V + \frac{2}{WT} SR + \frac{1}{WT} R^2 \quad .$$

The equations for the mean and variance of the reference channel are found by simply setting the signal terms to zero:

$$\mu^* = R$$

$$\sigma^{*2} = \frac{1}{WT} R^2 \quad .$$

Detection of a Lofar line is performed by comparing the narrowband in question to an average background that is near the narrowband frequency. The μ^*-equation assumes that the average background is caused entirely by broadband noise: there may be broadband target noise in the background but no Lofar lines. This is a reasonable assumption if the various lines are spread out enough on the display for there to be area visible on either side of any line. The value of σ^{*2} is not the variance of the average background. It is, instead, the variance of just one sample of background;

there may be many samples of background, all of them used to compute the average background value. The idea behind the σ^{*2}-equation is that if the signal sources were removed, then the data channel would have the same variance as the reference channel.

The detection threshold is assumed to be adequately described by:

$$a = \mu^* + d\,\sigma^* \quad .$$

The threshold is not a random variable in the sense that the output of data channel is a random variable; however, the threshold does change in response to the changing geometry and to the random process that generates the input values of source level, propagation loss, and noise level. The above definition of the detection threshold is equivalent to assuming a constant false alarm threshold. This means that if there is no Lofar line, then the random output from the data channel will exceed the threshold at a constant rate of:

$$f = \frac{n}{\Delta t} \frac{1}{\sqrt{2\pi}} \int_d^\infty e^{-\frac{1}{2}x^2} dx$$

where f = false alarm rate (number/min)

Δt = time step duration (min)

n = number of observations in a time step

d = number of reference channel standard deviations the threshold is set above the reference channel mean.

C. Broadband Statistics

The equations for μ and σ^* for the broadband case are derived by assuming that the amplitude A_{in} and phase θ_{in} from ith source in the nth frequency bin has the following statistics:

(1) $$E(A_{in}^2) = R'_{in}\ \Delta f$$

(2) $$Var(A_{in}^2) = [R'_{in}\ \Delta f]^2$$

(3) $$Cov(A_{in}^2, A_{im}^2) = 0 \qquad n \neq m$$

(4) $$E(\cos\theta_{ijn}\ \cos\theta_{ijm}) = 0 \qquad \begin{matrix} n \neq m \\ i \neq j \end{matrix}$$

where θ_{ijn} = $\theta_{in} - \theta_{jn}$; the difference of the ith and jth phase angles in the nth frequency bin.

Δf = $1/T$; width of the frequency bin (Hz).

T = averaging time (sec).

R'_{in} = mean square power from the ith source in the nth frequency bin of width Δf.

The first equation is a definition of symbols. The second equation is a Gaussian noise assumption. The third equation assumes that the noise power in one frequency bin is independent of the noise in another bin. The final equation assumes that the difference in phase angles between two sources is also independent from bin to bin.

The mean power from all frequency bins is:

$$E(P) = \sum_n \sum_i E(A_{in}^2)$$

$$E(P) = \sum_n \sum_i R'_{in}\ \Delta f$$

therefore

$$\mu = \int R'(f)\ df$$

where

$$R'(f) = \sum_i R'_i(f) \quad .$$

The sum over small frequency bins has been approximated by the integral over a power density function: $R'_i(f_n) \cong R'_{in}$, where f_n is the center frequency of the nth frequency bin.

The variance of the power is computed from the formulae derived in Section A. The variance of the power in the nth frequency bin is:

$$\mathrm{Var}(P_n) = \sum_i \mathrm{Var}(A_{in}^2) + \sum_{i \neq j}\sum E(A_{in}^2)\, E(A_{jn}^2)$$

$$\mathrm{Var}(P_n) = \sum_i [R'_{in}\ \Delta f]^2 + \sum_{i \neq j}\sum R'_{in}\ R'_{jn}\ \Delta f^2$$

$$\mathrm{Var}(P_n) = \left[\sum_i R'_{in}\right]^2 \frac{\Delta f}{T} \quad .$$

The covariance of the power in the nth bin with the power in the mth bin is zero because of the independence assumptions. Therefore, the variance of the total power is:

$$\mathrm{Var}(P) = \sum_i \mathrm{Var}(P_n)$$

$$\mathrm{Var}(P) = \frac{1}{T}\sum_n \left[\sum_i R'_{in}\right]^2 \Delta f$$

therefore
$$\sigma^2 = \frac{1}{T}\int [R'(f)]^2\, df$$

where
$$R'(f) = \sum_i R'_i(f) \quad .$$

The sum over small frequency bins has been approximated by the integration over a squared power density.

The equations for the broadband statistics for the data channel are:

$$\mu = \int R'\, df$$

$$\sigma^2 = \frac{1}{T}\int [R']^2\, df$$

where $R' = S' + N'$; and where S' is a sum of target spectra and N' is a sum of noise spectra. The equations for the broadband statistics for the reference channel are derived by assuming that the target spectra are reduced by a side lobe factor; the target spectra are not set to zero.

Thus, if S'^* is the sum of target spectra when all targets are in the side lobes, then the reference spectrum is:

$$R'^* = S'^* + N'$$

and the mean and variance of the reference channel is:

$$\mu = \int R'^* \, df$$

$$\sigma^2 = \frac{1}{T} \int [R'^*]^2 \, df \quad .$$

The reference channel statistics are defined this way so that the phenomenon of side lobe masking will be properly represented. The presence of a nearby target will drive the reference channel mean up and therefore the data channel output will have to satisfy a higher detection threshold. The net effect will be a low probability of detection when strong side lobe interference is present.

D. Modulated Broadband Statistics

The mean and variance of the data channel are derived by first assuming that the modulated voltage waveform from the array is given by:

$$x(t) = a(1 + m \cos \beta t) \cos \omega t \quad ,$$

where a is the amplitude of the carrier wave of angular frequency ω, and m is the modulation index of the modulating wave of angular frequency β. The modulated voltage can also be written as:

$$x(t) = \frac{ma}{2} \cos (\omega-\beta)t + a \cos \omega t + \frac{ma}{2} \cos (\omega+\beta)t \quad .$$

Since there is a band of waves with approximately the same amplitude a, the wave of amplitude ma/2 and angular frequency $\omega + \beta$ (the third term in the above equation) will interfere with a carrier wave of amplitude a and frequency $\omega + \beta$. The average power in the interference pattern is the sum of powers: $\frac{1}{2}a^2 + \frac{1}{2}\left(\frac{ma}{2}\right)^2$, because the two waves are incoherent. Another modulated wave of carrier frequency $\omega' = \omega + 2\beta$ will have a low side component of amplitude $ma/2$ and angular frequency $\omega' - \beta = \omega + \beta$. Thus the total power at frequency $f = (\omega+\beta)/2\pi$ is:

$$P'\,df = \frac{1}{2}a^2 + \frac{1}{2}\left(\frac{ma}{2}\right)^2 + \frac{1}{2}\left(\frac{ma}{2}\right)^2 \quad .$$

If the definitions: $M = m^2$ and $S'\,df = a^2/2$ are used, then the average signal power density at frequency f is:

$$P' = S' + \frac{1}{2}M\,S' \quad .$$

By assuming that the value of M changes slowly with frequency and that β is small compared with ω, the expression for the average power density can be integrated over a frequency band:

$$P = \int_{f_1}^{f_2} P'\,df \quad .$$

The integration limits, f_1 and f_2, must be large compared to $\beta/2\pi$.

The noise spectrum N' can be included in the integrand of the above equation because the noise is assumed to be independent of the signal. With the addition of noise, the mean power in the data channel is:

$$\mu = \int \left[S' + \frac{1}{2}MS' + N'\right] df \quad ,$$

and the mean of the reference channel is found by setting the modulation signal to zero:

$$\mu^* = \int [S' + N']\,df \quad .$$

The variances for the two channels are similar to the broadband case:

$$\sigma^2 = \frac{1}{T}\int \left[S' + \frac{1}{2}MS' + N'\right]^2 df$$

$$\sigma^{*2} = \frac{1}{T}\int [S' + N']^2\,df$$

where T is the averaging time.